Fully-Connected Tensor Network Decomposition and Its Application to Higher-Order Tensor Completion

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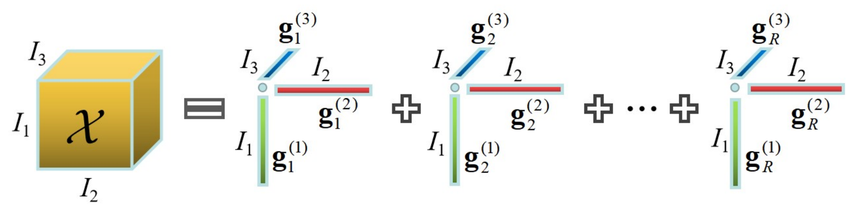
Background: Tensor Decomposition

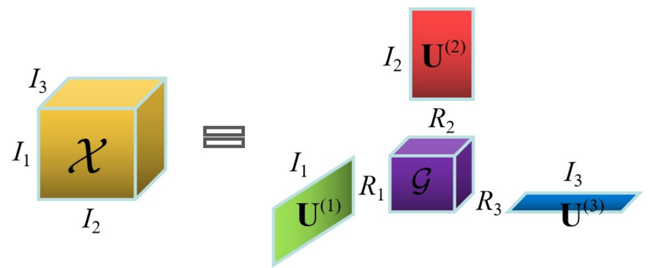
- Tensor decomposition aims to decompose a higher-order tensor to a set of low-dimensional factors and has powerful capability to capture the global correlations of tensors.
- CANDECOMP/PARAFAC (CP) decomposition:

$$\mathcal{X} = \sum_{r=1}^{R} \lambda_r \mathbf{g}_r^{(1)} \circ \mathbf{g}_r^{(2)} \circ \cdots \circ \mathbf{g}_r^{(N)}$$



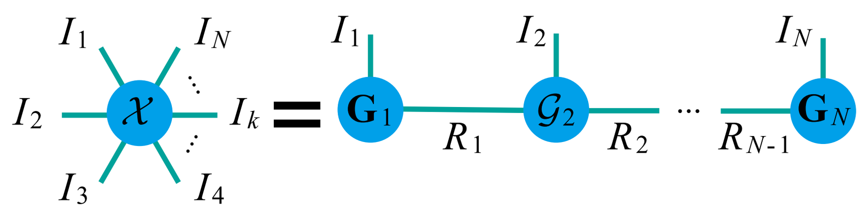
- Tucker decomposition:

$$\mathcal{X} = \mathcal{G} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \times_3 \cdots \times_N \mathbf{U}^{(N)}$$



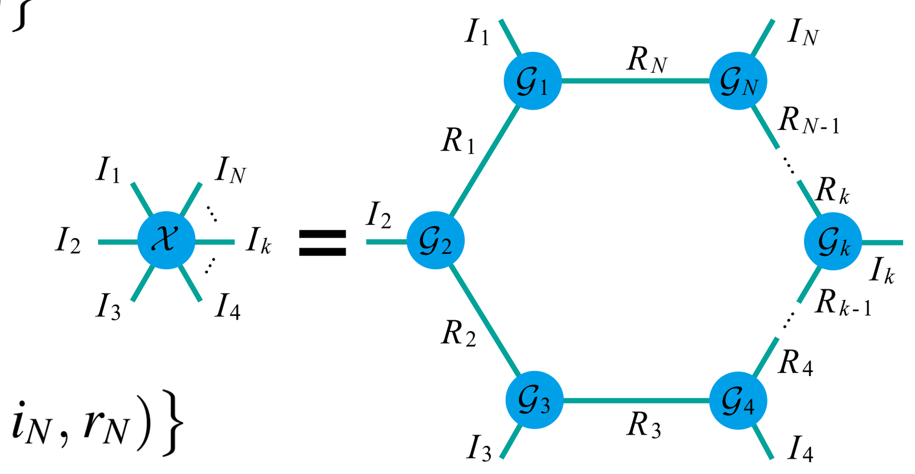
- Tensor train (TT) decomposition:

$$\mathcal{X}(i_1, i_2, \cdots, i_N) = \sum_{r_1=1}^{R_1}\sum_{r_2=1}^{R_2}\cdots\sum_{r_{N-1}=1}^{R_{N-1}} \{\mathbf{G}_1(i_1, r_1)\mathcal{G}_2(r_1, i_2, r_2)\cdots \mathbf{G}_N(r_{N-1}, i_N)\}$$



- Tensor ring (TR) decomposition:

$$\mathcal{X}(i_1, i_2, \cdots, i_N) = \sum_{r_1=1}^{R_1}\sum_{r_2=1}^{R_2}\cdots\sum_{r_N=1}^{R_N} \{\mathcal{G}_1(r_N, i_1, r_1)\mathcal{G}_2(r_1, i_2, r_2)\cdots \mathcal{G}_N(r_{N-1}, i_N, r_N)\}$$



Motivation

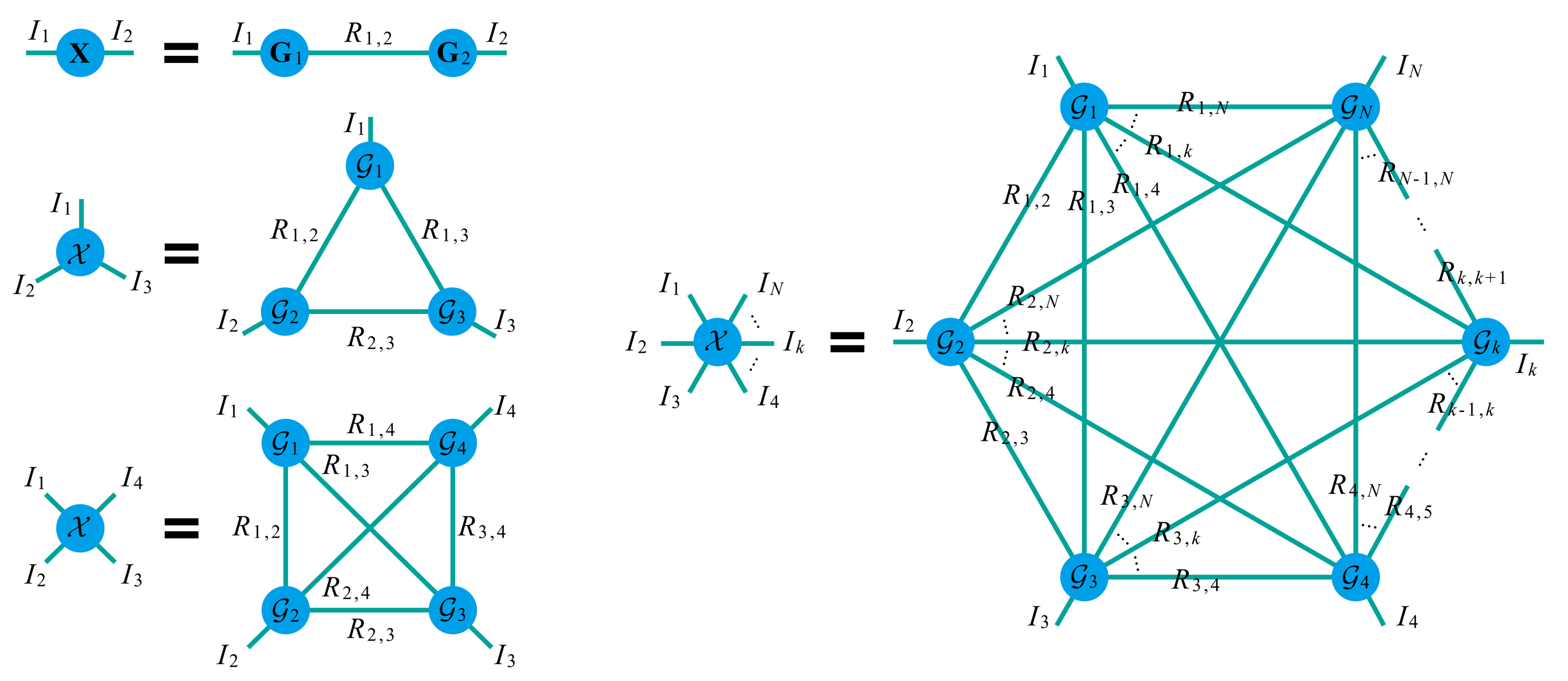
- CP decomposition faces difficulty in flexibly characterizing different correlations among different modes.
- Tucker decomposition only characterizes correlations among one mode and all the rest of modes, rather than between any two modes.
- TT and TR decompositions only establish a connection (operation) between adjacent two factors, rather than any two factors.
- TT and TR decompositions keep the invariance only when the tensor modes make a reverse permuting (TT and TR) or a circular shifting (only TR), rather than any permuting.

How to break through?

Fully-Connected Tensor Network Decomposition

- The fully-connected tensor network (FCTN) decomposition aims to decompose an Nth-order tensor $\mathcal{X}$ into a set of low-dimensional Nth-order factor tensors $\mathcal{G}_k$ ($k = 1, 2, \cdots, N$).

$$\mathcal{X}(i_1, i_2, \cdots, i_N) = \sum_{r_{1,2}=1}^{R_{1,2}}\sum_{r_{1,3}=1}^{R_{1,3}}\cdots\sum_{r_{1,N}=1}^{R_{1,N}}\sum_{r_{2,3}=1}^{R_{2,3}}\cdots\sum_{r_{2,N}=1}^{R_{2,N}}\cdots\sum_{r_{N-1,N}=1}^{R_{N-1,N}} \{\mathcal{G}_1(i_1, r_{1,2}, r_{1,3}, \cdots, r_{1,N}) \mathcal{G}_2(r_{1,2}, i_2, r_{2,3}, \cdots, r_{2,N})\cdots \mathcal{G}_k(r_{1,k}, r_{2,k}, \cdots, r_{k-1,k}, i_k, r_{k,k+1}, \cdots, r_{k,N})\cdots \mathcal{G}_N(r_{1,N}, r_{2,N}, \cdots, r_{N-1,N}, i_N)\}.$$



- characterizes the correlations between any two modes of tensors.
- has transpositional invariance, i.e.,

$$\mathcal{X} = \mathrm{FCTN}(\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_N) \Leftrightarrow \vec{\mathcal{X}}^{\mathbf{n}} = \mathrm{FCTN}(\vec{\mathcal{G}}^{\mathbf{n}}_{n_1}, \vec{\mathcal{G}}^{\mathbf{n}}_{n_2}, \cdots, \vec{\mathcal{G}}^{\mathbf{n}}_{n_N})$$

- can bound the rank of all generalized tensor unfolding.

FCTN Decomposition-Based TC Model

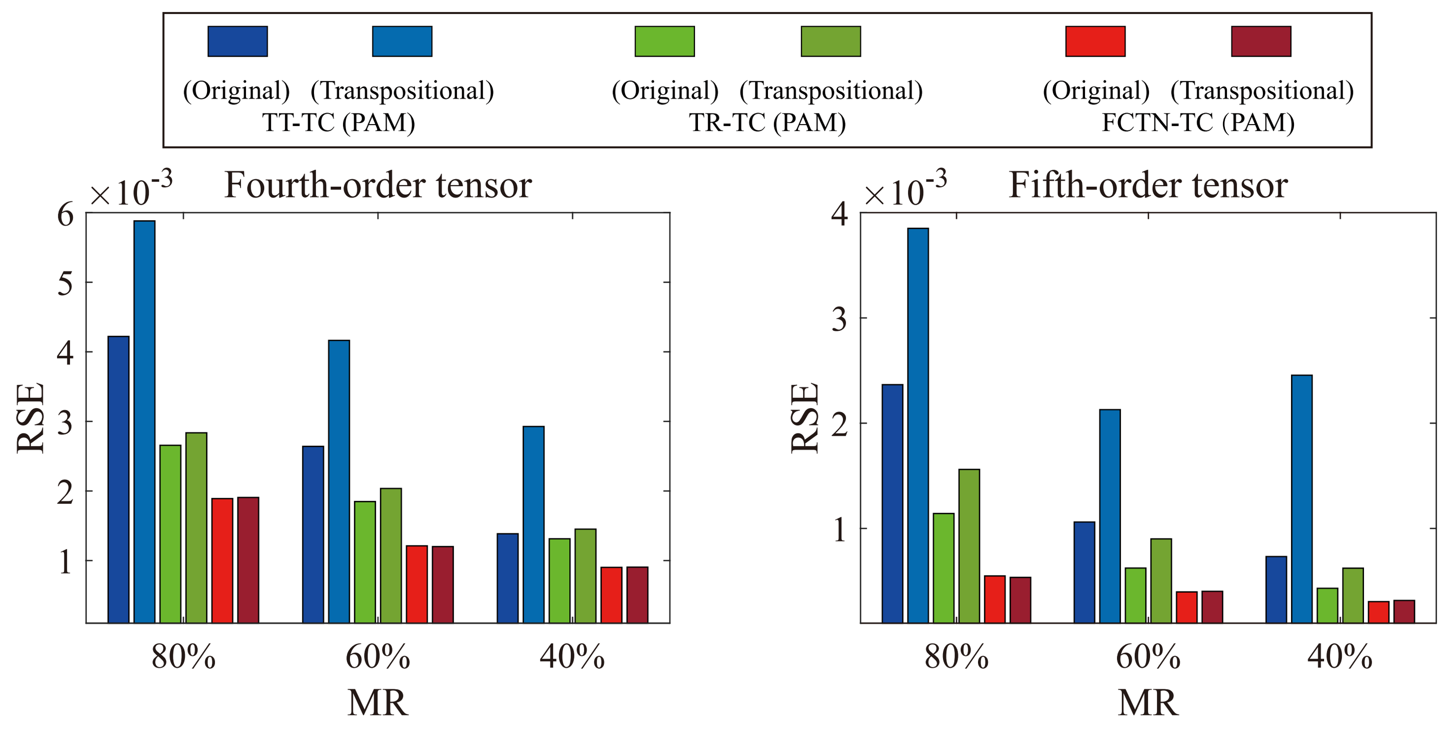
- Giving a partial observation $\mathcal{F}$ of the underlying tensor $\mathcal{X}$, the FCTN decomposition-based tensor completion (FCTN-TC) model is

$$\underset{\mathcal{X},\mathcal{G}}{\operatorname{argmin}} \ \frac{1}{2}\|\mathcal{X} - \mathrm{FCTN}(\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_N)\|_F^2, \quad \text{s.t. } \mathcal{P}_\Omega(\mathcal{X} - \mathcal{F}) = 0.$$

Experimental Results

RSE: relative error MR: missing ratio

- Synthetic Data Experiments (RSE)



- Color Video Data Experiments (PSNR)

Dataset	MR	95%	90%	80%	Mean time (s)	Dataset	MR	95%	90%	80%	Mean time (s)
news	Observed	8.7149	8.9503	9.4607	—	*containe*	Observed	4.5969	4.8315	5.3421	—
	HaLRTC	14.490	18.507	22.460	36.738		HaLRTC	18.617	21.556	25.191	34.528
	TMac	25.092	27.035	29.778	911.14		TMac	26.941	26.142	32.533	1224.4
	t-SVD	25.070	28.130	31.402	74.807		t-SVD	28.814	34.912	39.722	71.510
	TMacTT	24.699	27.492	31.546	465.75		TMacTT	28.139	31.282	37.088	450.70
	TRLRF	22.558	27.823	31.447	891.96		TRLRF	30.631	32.512	38.324	640.41
	FCTN-TC	**26.392**	**29.523**	**33.048**	473.50		FCTN-TC	**30.805**	**37.326**	**42.974**	412.72

- Traffic Data Experiments (RSE)